

# Complexity of distances, reducibility and universality

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The talk is based on the following papers.

- M. Cúth, M. Doucha and O. Kurka, *Complexity of distances between metric and Banach spaces: Theory of generalized analytic equivalence relations*, J. Math. Logic **23** (2023), 2250014.
- M. Cúth, M. Doucha and O. Kurka, *Complexity of distances between metric and Banach spaces: Reductions of distances between metric and Banach spaces*, Israel J. Math. **248** (2022), 383–439.
- O. Kurka, *Orbit pseudometrics and a universality property of the Gromov-Hausdorff distance*, Topology appl. **364** (2025), 109095.

## Definition

Let  $E$  and  $F$  be equivalence relations on Polish spaces  $X$  and  $Y$ . We say that  $E$  is *Borel reducible* to  $F$ , and write

$$E \leq_B F,$$

if there exists a Borel mapping  $f : X \rightarrow Y$  (so-called *reduction*) such that

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We say that  $E$  and  $F$  are *Borel bireducible* if  $E \leq_B F$  and  $F \leq_B E$ .

## Definition

By the *Urysohn space* we mean the (up to isometry) only complete separable metric space  $\mathbb{U}$  with the property that for any finite metric space  $A$  and any isometric embedding  $f : B \rightarrow \mathbb{U}$ , where  $B \subseteq A$ , there is an isometric embedding  $\tilde{f} : A \rightarrow \mathbb{U}$  extending  $f$ .

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## Definition

By the *Gurariy space* we mean the (up to isometry) only separable Banach space  $\mathbb{G}$  with the property that for every  $\varepsilon > 0$ , every finite-dimensional Banach spaces  $X$  and  $Y$  with  $X \subseteq Y$  and every linear isometry  $f : X \rightarrow \mathbb{G}$ , there exists an extension  $g : Y \rightarrow \mathbb{G}$  of  $f$  such that  $(1 + \varepsilon)^{-1}\|y\| \leq \|g(y)\| \leq (1 + \varepsilon)\|y\|$  for every  $y \in Y$ .

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It is known that the Gurariy space is isometrically universal for separable Banach spaces.

In the following definition, we introduce the Polish space of all complete separable metric spaces.

### Definition

We define

$$F(\mathbb{U}) = \{F \subseteq \mathbb{U} : F \text{ is closed}\},$$

and we equip  $F(\mathbb{U}) \setminus \{\emptyset\}$  with the *Wijsman topology*, defined as the coarsest topology for which the function

$$F \mapsto \text{dist}(u, F),$$

is continuous for each  $u \in \mathbb{U}$ .

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### Fact

$F(\mathbb{U}) \setminus \{\emptyset\}$  equipped with the *Wijsman topology* forms a Polish space.

In an analogous way, we introduce the Polish space of all separable Banach spaces.

### Definition

We define

$$\mathcal{SE}(\mathbb{G}) = \{F \subseteq \mathbb{G} : F \text{ is closed and linear}\}.$$

with the corresponding Wijsman topology.

In an analogous way, we introduce the Polish space of all separable Banach spaces.

### Definition

We define

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with the corresponding Wijsman topology.

With these two codings, we can consider equivalence relations between metric/Banach spaces as equivalence relations on a Polish space.

## Definition

Let  $G$  be a group with identity  $e$  and let  $X$  be a set. By a *group action*  $G \curvearrowright X$  we mean a mapping  $(g, x) \in G \times X \mapsto g \cdot x \in X$  satisfying

$$e \cdot x = x$$

and

$$(gh) \cdot x = g \cdot (h \cdot x).$$

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## Definition

Let  $G \curvearrowright X$  be a Borel action of a Polish group  $G$  on a Polish space  $X$ . The corresponding *orbit equivalence relation* is defined by

$$xE_G^X y \iff \exists g \in G : g \cdot x = y.$$

## Theorem (Miller, 1977)

*The equivalence classes of  $E_G^X$  are Borel.*

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The universal orbit equivalence is also bireducible with:

- the linear isometry relation of separable Banach spaces (Melleray),
- the affine homeomorphism of Choquet simplices (Sabok),
- the isomorphism relation of separable  $C^*$ -algebras (Sabok),
- the homeomorphism relation of compact metric spaces (Zielinski).

Consequently, all these relations have Borel equivalence classes.

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## Corollary

*The relation  $E_1$  is not Borel reducible to the isometry relation of complete separable metric spaces (as well as to the other equivalences from the previous slide).*

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## Corollary

*The relation  $E_1$  is not Borel reducible to the isometry relation of complete separable metric spaces (as well as to the other equivalences from the previous slide).*

The following question is an important open problem.

## Question

Let  $E$  be a Borel equivalence relation into which  $E_1$  is not Borel reducible. Is  $E$  Borel reducible to an orbit equivalence relation then?

## Definition

The *Gromov-Hausdorff distance* of non-empty metric spaces  $M$  and  $N$  is defined by

$$\varrho_{GH}(M, N) = \inf_{\substack{X \text{ metric space} \\ i_M: M \hookrightarrow X \\ i_N: N \hookrightarrow X}} \varrho_H^X(i_M(M), i_N(N))$$

(where  $i_M, i_N$  are isometric embeddings of  $M, N$  into  $X$  and  $\varrho_H^X$  denotes the Hausdorff distance between subsets of  $X$ ).

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## Definition

The *Kadets distance* of Banach spaces  $X$  and  $Y$  is defined by

$$\varrho_K(X, Y) = \inf_{\substack{Z \text{ Banach space} \\ i_X: X \hookrightarrow Z \\ i_Y: Y \hookrightarrow Z}} \varrho_H^Z(i_X(B_X), i_Y(B_Y))$$

(where  $i_X, i_Y$  are linear isometric embeddings of  $X, Y$  into  $Z$ ).

## Definition

The *Banach-Mazur distance* of Banach spaces  $X$  and  $Y$  is defined by

$$d_{BM}(X, Y) = \inf \{ \log \|T\| \|T^{-1}\| : T \in \mathcal{L}(X, Y) \text{ is an isomorphism} \}$$

(so  $d_{BM}(X, Y) = \infty$  if  $X$  and  $Y$  are not isomorphic).

## Definition

If  $\varrho : X \times X \rightarrow [0, \infty]$  is a pseudometric on a set  $X$ , we define

$$E_\varrho = \{(x, y) : \varrho(x, y) = 0\}.$$

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## Theorem (Ben Yaacov, Doucha, Nies, Tsankov)

*The equivalence classes of the relations  $E_{\varrho_{GH}}$  and  $E_{\varrho_K}$  are Borel.*

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It was pointed out by C. Rosendal that any orbit equivalence is reducible to both  $E_{\varrho_{GH}}$  and  $E_{\varrho_K}$ . The following remains open.

## Question (Ben Yaacov, Doucha, Nies, Tsankov)

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Later, we will give a negative answer to the following question.

## Question (Ben Yaacov, Doucha, Nies, Tsankov)

Is  $\varrho_{GH}(M, \cdot)$  a Borel function for every  $M \in F(\mathbb{U}) \setminus \emptyset$ ?  
Is  $\varrho_K(X, \cdot)$  a Borel function for every  $X \in \mathcal{SE}(\mathbb{G})$ ?

## Theorem 1 (Cúth, Doucha, K.)

*The following equivalences are mutually Borel bireducible:*

- $E_{\varrho_{GH}}$
- $E_{\varrho_{GH}}$  restricted to metric spaces with distances in  $\{0\} \cup [1, 2]$
- $E_{\varrho_K}$
- $E_{\varrho_{BM}}$
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- $E_{\varrho_K}$
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## Corollary

The equivalence classes of the relation  $E_{\varrho_{BM}}$  are Borel.

## Definition (Cúth, Doucha, K.)

Let  $\varrho_X$  and  $\varrho_Y$  be pseudometrics on Polish spaces  $X$  and  $Y$ . We say that  $\varrho_X$  is *Borel-u.c. reducible* to  $\varrho_Y$ , and write

$$\varrho_X \leq_{B,u} \varrho_Y,$$

if there exists a Borel mapping  $f : X \rightarrow Y$  such that, for every  $\varepsilon > 0$ , there are  $\delta_X > 0$  and  $\delta_Y > 0$  satisfying

$$\forall x, y \in X : \quad \varrho_X(x, y) < \delta_X \Rightarrow \varrho_Y(f(x), f(y)) < \varepsilon$$

and

$$\forall x, y \in X : \quad \varrho_Y(f(x), f(y)) < \delta_Y \Rightarrow \varrho_X(x, y) < \varepsilon.$$

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We say that  $\varrho_X$  is *Borel-u.c. bireducible* with  $\varrho_Y$  if  $\varrho_X \leq_{B,u} \varrho_Y$  and  $\varrho_Y \leq_{B,u} \varrho_X$ .

## Remark

The reducibility between pseudometrics is

- a strengthening of the reducibility between equivalences in the sense that

$$\varrho \leq_{B,u} \varrho' \quad \Rightarrow \quad E_{\varrho} \leq_B E_{\varrho'},$$

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- a strengthening of the reducibility between equivalences in the sense that

$$\varrho \leq_{B,u} \varrho' \quad \Rightarrow \quad E_{\varrho} \leq_B E_{\varrho'},$$

- a generalization of the reducibility between equivalences in the sense that

$$\varrho_E \leq_{B,u} \varrho_F \quad \Leftrightarrow \quad E \leq_B F.$$

Here, for an equivalence relation  $E$  on a Polish space  $X$ , the pseudometric  $\varrho_E$  is defined by

$$\varrho_E = \begin{cases} 0, & x E y, \\ 1, & \text{otherwise.} \end{cases}$$

The proof of Theorem 1 provides also a quantitative version:

### Theorem 2 (Cúth, Douča, K.)

*The following distances are mutually Borel-u.c. bireducible:*

- $\varrho_{GH}$
- $\varrho_{GH}$  restricted to metric spaces with distances in  $\{0\} \cup [1, 2]$
- $\varrho_K$
- $\varrho_{BM}$
- ...

### Theorem 3 (Cúth, Douča, K.)

If  $\varrho$  is a pseudometric such that

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then the function  $\varrho(x, \cdot)$  is not Borel for some  $x$ .

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If  $\varrho$  is a pseudometric such that

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### Corollary

If  $\varrho$  is any of the distances  $\varrho_{GH}, \varrho_K, \varrho_{BM}, \dots$ , then the function  $\varrho(x, \cdot)$  is not Borel for some  $x$ .

By  $\mathcal{P}(\mathbb{N})$  we denote the space of all subsets of  $\mathbb{N}$  (which is actually nothing else than a copy of the Cantor space  $2^{\mathbb{N}}$ ). By  $K(\mathcal{P}(\mathbb{N}))$  we mean the hyperspace of all compact subsets of  $\mathcal{P}(\mathbb{N})$ .

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Theorem 3 then follows from the next proposition and the classical Hurewicz's result that the set of all  $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$  containing an infinite set is a complete analytic set, and so it is not Borel.

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### Proposition

*Let us consider the space*

$$X = \left( \bigoplus G_n \right)_{\ell_1},$$

*where  $G_1, G_2, \dots$  is a dense sequence of finite-dimensional spaces.*

*Then, for every  $\varepsilon > 0$ , there exists a Borel mapping*

*$\mathfrak{S} : K(\mathcal{P}(\mathbb{N})) \rightarrow \mathcal{SE}(\mathbb{G})$  such that*

*(a) if  $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$  contains an infinite set, then*

$$\rho_{BM}(\mathfrak{S}(\mathcal{A}), X) \leq \varepsilon,$$

*(b) if  $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$  consists of finite sets only, then  $\mathfrak{S}(\mathcal{A})$  and  $X$  are not isomorphic, i.e.,  $\rho_{BM}(\mathfrak{S}(\mathcal{A}), X) = \infty$ .*

## Proof of Proposition (sketch).

For  $\mathcal{A} \in K(\mathcal{P}(\mathbb{N}))$  and  $0 < \theta < 1$ , a Tsirelson type space  $T[\mathcal{A}, \theta]$  of Argyros and Deliyanni is defined as the completion of  $c_{00}$  under the implicitly defined norm

$$\|x\|_{\mathcal{A}, \theta} = \max \left\{ \|x\|_{\infty}, \theta \sup \sum_{k=1}^n \|E_k x\|_{\mathcal{A}, \theta} \right\},$$

where the “sup” is taken over all finite families  $\{E_1, \dots, E_n\}$  of finite subsets of  $\mathbb{N}$  such that

$$\exists A \in \mathcal{A} \exists m_1, \dots, m_n \in A : m_1 \leq E_1 < m_2 \leq E_2 < \dots < m_n \leq E_n.$$

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Given  $\varepsilon > 0$ , we put  $\theta = e^{-\varepsilon}$  and

$$X_{\mathcal{A}} = T[\mathcal{A}_1, \theta] \oplus_1 X, \quad \mathcal{A} \in K(\mathcal{P}(\mathbb{N})),$$

where  $\mathcal{A}_1 = \{A \cup \{1\} : A \in \mathcal{A}\}$ . Then  $X_{\mathcal{A}}$  satisfies the requirements (a) and (b) on  $\mathfrak{S}(\mathcal{A})$ . □

A natural class of pseudometrics is obtained by a generalization of orbit equivalence relations:

### Definition

Let  $G \curvearrowright X$  be a Borel action of a Polish group  $G$  on a Polish space  $X$ , and let  $d$  be a pseudometric on  $X$  with the property

$$d(x, y) = d(g \cdot x, g \cdot y), \quad x, y \in X, g \in G.$$

Then we define

$$\varrho_{G,d}(x, y) = \inf\{d(g \cdot x, y) : g \in G\}$$

and call such pseudometric an *orbit pseudometric*.

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and call such pseudometric an *orbit pseudometric*.

### Remark

In fact, every pseudometric is an orbit pseudometric if we consider a trivial action. For this reason, one should impose some restrictions.

## Example

If  $d$  is the discrete metric, that is,

$$d(x, y) = \begin{cases} 0, & x = y, \\ 1, & x \neq y, \end{cases}$$

then

$$\varrho_{G,d}(x, y) = \begin{cases} 0, & \exists g \in G : g \cdot x = y, \\ 1, & \text{otherwise.} \end{cases}$$

So,  $E_{\varrho_{G,d}}$  is nothing else but the orbit equivalence  $E_G^X$ .

## Example

Let  $X = [1, 2]^{\mathbb{N}^2}$ , let  $S_\infty$  denote the group of permutations of  $\mathbb{N}$ , and let

$$(\pi \cdot x)(m, n) = x(\pi^{-1}(m), \pi^{-1}(n))$$

and

$$d_2(x, y) = \sup_{m \neq n} |x(m, n) - y(m, n)|$$

for  $\pi \in S_\infty, x, y \in X, m, n \in \mathbb{N}, m \neq n$ .

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It is not difficult to show that  $\rho_{S_\infty, d_2}(x, y) = 2\rho_{GH}((\mathbb{N}, x), (\mathbb{N}, y))$  whenever one of the sides is less than 1. It follows:

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### Theorem (Cúth, Douča, K.)

$\varrho_{S_\infty, d_2}$  is Borel-u.c. bireducible with  $\varrho_{GH}$ .

Using a method of G. Hjorth, we prove the following generalization of the result of Kechris and Louveau.

#### Theorem 4 (Cúth, Doucha, K.)

*Let  $G, X$  and  $d$  be as above, and let the action  $G \curvearrowright X$  be continuous. Moreover, let  $d$  be analytic (in the sense that  $\{(x, y) \in X^2 : d(x, y) < \alpha\}$  is analytic for each  $\alpha$ ).*

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*If  $d$  is a complete metric and generates a topology that is finer than the original topology of  $X$ , then  $E_1$  is not Borel reducible to  $E_{\varrho_{G,d}}$ .*

Considering the result from the previous slide, we obtain:

### Corollary

$E_1$  is not Borel reducible to  $E_{\varrho_{GH}}, E_{\varrho_K}, E_{\varrho_{BM}}, \dots$

Next, we show that the Gromov-Hausdorff distance is universal for a certain class of orbit pseudometrics.

### Theorem 5 (K.)

*Let  $G$  be a Polish group acting continuously on a Polish space  $X$ . Let  $d$  be a lower semicontinuous pseudometric on  $X$  such that  $d(x, y) = d(gx, gy)$  for any  $x, y \in X$  and  $g \in G$ .*

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### Corollary

Let  $\varrho_{G,d}$  be as above. Then  $E_{\varrho_{G,d}}$  is Borel reducible to  $E_{GH}$ , and so its equivalence classes are Borel and  $E_1$  is not Borel reducible to it.

We now present some basic tools and ideas of the proof of Theorem 5.

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We need the following result.

### Theorem (Melleray)

*Let  $X$  be a complete separable metric space of diameter at most 1 and let  $G$  be a closed subgroup of  $\text{Iso}(X)$ . Then there exists an extension  $Y$  of  $X$  such that*

- *$Y$  is a complete separable metric space,*
- *any member of  $G$  can be extended in a unique way to a surjective isometry on  $Y$ ,*
- *any surjective isometry on  $Y$  is an extension of a member of  $G$ .*

Without loss of generality,  $d \leq 1$ .

We can also assume that the sequence  $s_1, s_2, \dots$  is non-decreasing.

Let  $\gamma$  be a compatible right-invariant metric on  $G$  with  $\gamma \leq 1$ .

Let  $\delta_X$  be a compatible complete metric on  $X$  with  $\delta_X \leq 1$ .

Let us consider the maximum distance on  $G \times X$ .

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Then Melleray's theorem provides:

### Claim

*There is an extension  $Z$  of  $G \times X$  such that*

- *$Z$  is a complete separable metric space of diameter at most 2,*
- *for any  $h \in G$ , the isometry  $I_h : (g, x) \mapsto (gh, x)$  can be extended in a unique way from  $G \times X$  to a surjective isometry on  $Z$ ,*
- *any surjective isometry on  $Z$  is an extension of  $I_h$  for some  $h \in G$ .*

We define

$$W = (Z \times \mathbb{N}) \cup (G \times X \times [0, 1] \times \mathbb{N})$$

and a metric  $m$  on  $W$  as follows:

$$m((z_1, k_1), (z_2, k_2)) = 100 \cdot |2^{k_1} - 2^{k_2}| + 2^{\min\{k_1, k_2\}} \delta_Z(z_1, z_2),$$

$$m((z, l), (g, x, u, k)) = u + 10 \cdot 2^k + m((z, l), ((g, x), k)),$$

$$m((g_1, x_1, u_1, k), (g_2, x_2, u_2, k)) = |u_1 - u_2| + 2^k \delta_Z((g_1, x_1), (g_2, x_2)),$$

and for  $k_1 \neq k_2$ ,

$$m((g_1, x_1, u_1, k_1), (g_2, x_2, u_2, k_2)) = u_1 + 10 \cdot 2^{k_1} + u_2 + 10 \cdot 2^{k_2} \\ + m(((g_1, x_1), k_1), ((g_2, x_2), k_2)).$$

For every  $p \in X$ , let us consider the subspace of  $W$

$$W_p = (Z \times \mathbb{N}) \cup \{(g, x, s_k(gp, x), k) : g \in G, x \in X, k \in \mathbb{N}\}.$$

Let  $Y$  be the completion of  $W$  and, for every  $p \in X$ , let  $Y_p$  be the closure of  $W_p$  in  $Y$ .

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*The mapping  $p \mapsto Y_p$  from  $X$  to  $F(Y)$  is Borel.*

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### Claim

For all  $p, q \in X$ ,

$$\varrho_{Iso(Y), \varrho_H}(Y_p, Y_q) \leq \varrho_{G, d}(p, q) \leq 2\varrho_{GH}(Y_p, Y_q).$$

The methods developed for proving Theorem 5 give also:

### Theorem 6 (K.)

*The Gromov-Hausdorff distance  $\varrho_{GH}$  is Borel-u.c. bireducible with the orbit pseudometric  $\varrho_{Iso(\mathbb{U}), \varrho_H}$  on  $F(\mathbb{U}) \setminus \{\emptyset\}$ .*

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With some additional work involving the certain renormings of  $\ell_2$ , it is possible to show the Banach spaces version:

### Theorem 7 (K.)

*The Kadets distance  $\varrho_K$  is Borel-u.c. bireducible with the orbit pseudometric  $\varrho_{Iso_L(\mathbb{G}), \tilde{\varrho}_H}$  on  $\mathcal{SE}(\mathbb{G})$ .*

Here, we denote

$$\tilde{\varrho}_H(E, F) = \varrho_H(B_E, B_F).$$

### Theorem (Ferenczi, Louveau, Rosendal, 2009)

*The relation of isomorphism between separable Banach spaces is a complete analytic equivalence relation. (More precisely, any analytic equivalence relation is Borel reducible to it.)*

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### Question

Is there a natural example of a complete analytic pseudometric?

Let us recall that it is open if  $E_{\varrho_{GH}}$  is reducible to an orbit equivalence.

This is unclear even for a considerably simplified version of  $E_{\varrho_{GH}}$ .

### Question

Is the equivalence relation

$$\{(x, y) \in [0, 1]^{\mathbb{Z}} : (\forall \varepsilon > 0 \exists k \in \mathbb{Z} \forall l \in \mathbb{Z} : |x(l+k) - y(l)| < \varepsilon)\}$$

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This is related to the problem if  $E_1$  is a minimal Borel equivalence that is not reducible to an orbit equivalence relation.

Indeed, if  $E_1$  is minimal, then the answer to the question must be positive.



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